Log Calabi-Yau geometry and Cremona maps

I - Cremona groups and the Sarkisov Program

$\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$

$\mathrm{Bir}(\mathbb{P}^n) := \{ f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \text{ bir} \}$, Cremona

Minimal Model Program (MMP)

$\mathbb{P}^n \longrightarrow \mathrm{Spec}(\mathbb{C})$ is a Mori fibered space

Rough sketch of the MMP

$$
X \dashrightarrow \begin{cases} \cdot\ X' \text{ with } K_{X'} \text{ nef (minimal model)}, & \text{if } \kappa(X) \geq 0 \\ \cdot\ \begin{array}{c} X' \\ \downarrow \\ S \end{array} \text{ Mori fibered space}, & \text{if } \kappa(X) = -\infty \end{cases}
$$

proj. var. with term. sing.

MMP holds for dim ≤ 3

MMP is conjectural in some cases for dim > 3

Def.: A Mori fibered space (MFS) is a normal proj. var. X together with a morphism $f: X \to S$, where $\dim X > \dim S$, st

1) $f_* \mathcal{O}_X = \mathcal{O}_S$
2) $-K_X$ is f-ample, and
3) $\rho(X/S) = \rho(X) - \rho(S) = 1$

We denote such a structure by X/S.

Thm (Sarkisov Program - Corti 1995; Hacon, McKernan 2013):

$$\begin{tikzcd}
X_0 \arrow[r, dashed] \arrow[d] \arrow[rrrr, dashed, bend left, "\phi \text{ bir}"] \arrow[dr, phantom, "\text{Sark. link}"] & X_1 \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\text{Sark. link}"] & \cdots \arrow[r, dashed] & X_{n-1} \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\text{Sark. link}"] & X_n \arrow[d] \\
S_0 & S_1 & & S_{n-1} & S_n
\end{tikzcd}$$

II - Log Calabi-Yau geometry

Def.: A log Calabi-Yau (CY) pair is a lc (X, D) pair consisting of a normal proj. var. and a reduced Weil divisor D st

$$K_X + D \sim_{\mathbb{Z}} 0.$$

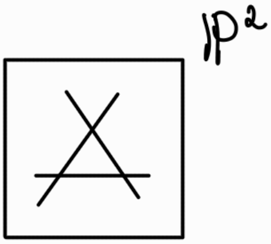
Rmk.: $n = \dim X$

(X, D) CY pair $\Rightarrow \exists\ \omega := \omega_{X,D} \in \Omega^n_X$ unique up to nonzero scaling st

$$D + \operatorname{div}(\omega) = 0.$$

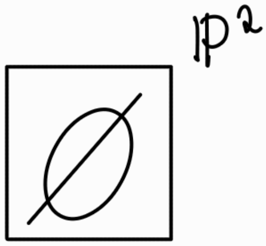
We call this ω the volume form.

Ex.: $X = \mathbb{P}^2$



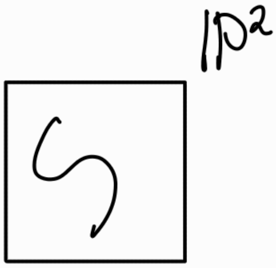
$L_1 + L_2 + L_3$

3 pairwise concurrent lines



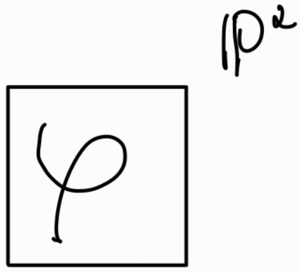
$L + C$

line and conic



C

nonsingular cubic



C

nodal cubic

Def.: $f : (X, D_X) \overset{\text{bir}}{\dashrightarrow} (Y, D_Y)$ CY pairs.
f is vol. pres. if $f^* \omega_{Y,D_Y} = \lambda\, \omega_{X,D_X}$, for some $\lambda \in \mathbb{C}^*$.

Rmk.: $\mathrm{Bir}^{vp}(X, D) < \mathrm{Bir}(X)$

$\mathrm{Bir}^{vp}(X, D)$: group of self-vol. pres. maps; $<$: subgroup

- (X, D) CY pair

$K_X + D \sim 0 \Rightarrow -K_X = D \geq 0$

$\Rightarrow K_X$ is not pseudoeffective

$\overset{(*)}{\Rightarrow} X$ is uniruled

$\Rightarrow \kappa(X) = -\infty$

$\Rightarrow$ output of the MMP on X is a MFS X'/S

(*) Boucksom, Demailly, Păun, Peternell Thm

- (X, D) is a minimal model for the log MMP, since $K_X + D$ is nef

Thm (Vol. pres. Sark. Program - Corti, Kaloghiros 2016):

$$\begin{tikzcd}[column sep=small]
(X_0, D_0) \arrow[r, dashed, "\text{vol. pres. Sark. link}"'] \arrow[d] \arrow[rrrr, dashed, bend left, "\varphi \text{ vol. pres.}"] & (X_1, D_1) \arrow[r, dashed, "\text{vol. pres. Sark. link.}"'] \arrow[d] & \cdots \arrow[r, dashed] & (X_{n-1}, D_{n-1}) \arrow[r, dashed, "\text{vol. pres. Sark. link}"'] \arrow[d] & (X_n, D_n) \arrow[d] \\
S_0 & S_1 & & S_{n-1} & S_n
\end{tikzcd}$$

Def.: Let X be a proj. var. and $Y \subset X$ an irred. subvar.. The decomposition group of Y in $\mathrm{Bir}(X)$ is the group

$$\mathrm{Bir}(X,Y) = \{ f : X \overset{\text{bir}}{\dashrightarrow} X \mid f|_Y : Y \overset{\text{bir}}{\dashrightarrow} Y \}$$

Not.: $X = \mathbb{P}^n$, $\mathrm{Bir}(X,Y) = \mathrm{Dec}(Y)$

$$\mathrm{Bir}(X,Y) = \mathrm{Bir}^{vp}(X,Y)$$

Prop. 2.6 (Araujo, Corti, Massarenti 2024)

Y is a prime divisor
(X,Y) is a can CY pair

III - 2-dimensional case

$C \subset \mathbb{P}^2$ nonsingular cubic

$(\mathbb{P}^2, C)$ is a can. CY pair

Thm (Pan 2007): Let $C \subset \mathbb{P}^2$ be an irred., nonsingular, nonrational curve. Assume $\exists\, \phi \in \mathrm{Dec}(C) \setminus PGL(3,\mathbb{C})$. Then $\deg(C) = 3$ and $\mathrm{Bs}(\phi) \subset C$.

Lemma (- 2023): $C \subset \mathbb{P}^2$ nonsingular cubic, $\phi \in \mathrm{Dec}(C) \setminus PGL(3,\mathbb{C})$. Then $\underline{\mathrm{Bs}}(\phi) \subset C$.

Thm (- 2023): $C \subset \mathbb{P}^2$ nonsingular cubic. The standard Sarkisov Program applied to an el. of $\mathrm{Dec}(C)$ is automatically vol. pres.

IV - 3-dimensional case

$D \subset \mathbb{P}^3$ can. quartic surface

$\Rightarrow D$ is either nonsingular or has ADE sing.

$(\mathbb{P}^3, D)$ is a can CY pair

$$
\begin{tikzcd}
 & E \arrow[dl, leftrightarrow] & Z \arrow[dl, "\text{div. ext.}"] & & \\
C \ni P & \mathbb{P}^3 \arrow[rrr, dashed, "\phi \text{ bir}"] \arrow[d] & & & X \arrow[d] \\
 & \mathrm{Spec}(\mathbb{C}) & & & S
\end{tikzcd}
$$

If $\pi(E) = \{P\}$, by Kawakita's Thm, $\exists$ an analytic neighborhood of P st π is equiv. to a $(1, a, b)$-weighted blowup at P, where $GCD(a, b) = 1$.

$$\pi \text{ "=" } Bl_{(1,a,b)}$$

Thm (Guerreiro 2022): Assume $\pi = Bl_{(1,a,b)}$

π initiates a ~~Sark.~~ link $\iff (a,b) \in \{(1,1), (1,2), (2,3), (2,5)\}$,

up to permutation

$$
\begin{tikzcd}
 & E \arrow[dl, leftrightarrow] & (Z, D_Z) \arrow[dl, "\substack{\text{vol. pres.}\\ \text{div. ext.}}"] & \\
C \text{ on } P & (\mathbb{P}^3, D) \arrow[rr, dashed, "\phi \text{ vol. pres.}"] \arrow[d] & & (X, D_X) \arrow[d] \\
 & \operatorname{Spec}(\mathbb{C}) & & S
\end{tikzcd}
$$

$$
\left.\begin{array}{l} \pi(E) = \{P\} \\ \pi \text{ vol. pres.} \end{array}\right\} \Rightarrow P \in \operatorname{Sing}(D)
$$

$$\pi \text{ ``$=$'' } Bl_{(1,1,1)} \Rightarrow \pi = Bl_{(1,1,1)}$$

$$\pi \text{ ``$=$'' } Bl_{(1,1,2)} \Rightarrow \pi = Bl_{(1,1,2)}$$

locally

Thm (-2023): Let $(\mathbb{P}^3, D)$ be a CY of cor. 2 and $\pi: (X, \tilde{D}) \longrightarrow (\mathbb{P}^3, D)$ be a vol. pres. toric $(1,a,b)$-weighted blowup of a torus invariant pt.. Then this pt. is a sing. of D and, up to perm., the only possibilities for the weights initiating a vol. pres. Sarkisov link are described in the following table.

type of sing.	possible vol. pres. weights
A_1	$(1,1,1)$
A_2	$(1,1,1)$, $(1,1,2)$
A_3	$(1,1,1)$, $(1,1,2)$
A_4	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$
A_5	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$
$A_{\geq 6}$	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$, $(1,2,5)$
D_4	$(1,1,1)$, $(1,1,2)$
$D_{\geq 5}$	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$
E_6	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$
E_7	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$
E_8	$(1,1,1)$, $(1,1,2)$, $(1,2,3)$